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ASYMPTOTIC SOLUTION OF THE PROBLEM OF THE ACTION OF A STAMP ON AN ELASTIC LAYER LYING ON THE SURFACE OF A COMPRESSIBLE FLUID OF INFINITE DEPTH

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This paper considers a two-dimensional linear unsteady problem of rigid-stamp indentation on an elastic layer of finite thickness lying on the surface of a compressible fluid of infinite depth. The Lamé equations holds for the elastic layer, and the wave equation for the fluid velocity potential. Using the Laplace and Fourier transforms, the problem is reduced to determining the contact stresses under the stamp from the solution of an integral equation of the first kind, whose kernel has a logarithmic singularity. An asymptotic solution of the problem is constructed for large times of interaction.

Key words: *stamp, elastic layer, compressible fluid, contact stresses.*

Introduction. For operation of facilities located on ice, it is necessary to know the dynamic loads caused by the motion of these objects, for example, under the action of vibration or an applied load. An ice sheet is often modeled by a thin elastic plate floating on a fluid surface [1, 2]. It is assumed in this case that ice completely covers the fluid free surface and is loaded by a concentrated force or a locally distributed time-periodic pressure on its surface. Another model is based on the assumption that ice is an elastic half-space, on whose surface there is a rigid stamp, which models the facility. From the state of rest, the stamp begins to move in a predetermined manner, and the effect of the fluid is ignored [3, 4]. Because ice has finite thickness, it is of interest to solve the problem of a stamp on an elastic layer lying on a fluid surface. In the present work, an asymptotic solution of this hydroelastic problem was obtained in a linear formulation for large times of interaction.

Formulation of the Problem. A rigid stamp of width $2a$ ($-a \leqslant x_1 \leqslant a$) is pressed in a predetermined manner into an elastic layer of thickness $2h_1$ ($-\infty < x_1 < \infty$ and $-h_1 \leqslant y_1 \leqslant h_1$) which lies on the surface of a compressible fluid of infinite depth ($-\infty < y_1 \leqslant -h_1$). If the elastic medium and the fluid are assumed to be in the state of rest before the stamp indentation ($t_1 < 0$), then the fluid flow is potential and, at the initial time $t_1 = 0$, the displacements of the elastic medium u_1 and v_1, the velocities of these displacement, and the fluid flow potential φ_1 and $\partial\varphi_1/\partial t_1$ are equal to zero. In the zone of contact of the stamp with the upper boundary of the layer $y_1 = h_1$, friction and adhesion forces are ignored and, outside the stamp, the upper boundary of the layer is not loaded. The lower boundary of the layer is acted upon only by the normal stresses caused by the fluid flow, and the condition of contact of the fluid and the elastic layer is satisfied. The shape of the stamp and the law of stamp indentation into the layer are defined by the function $f(x_1, t_1)$. As $(x_1, y_1) \to \infty$, displacements and stresses are absent.

Under the assumptions made above, the problem reduces to solving the Lamé equations in the elastic layer with respect to the dimensionless displacements u and v [5]:

$$\frac{\partial^2 u}{\partial x^2} + (1-\beta_1^2)\,\frac{\partial^2 v}{\partial x\,\partial y} + \beta_1^2\,\frac{\partial^2 u}{\partial y^2} = \beta_1^2\,\frac{\partial^2 u}{\partial t^2}; \qquad (1)$$

$$\beta_1^2\,\frac{\partial^2 v}{\partial x^2} + (1-\beta_1^2)\,\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 v}{\partial y^2} = \beta_1^2\,\frac{\partial^2 v}{\partial t^2}. \qquad (2)$$

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Here $\beta_1 = c_2/c_1$, $c_1^2 = (\lambda + 2\mu)/\rho$, $c_2^2 = \mu/\rho$ are the velocities of propagation of longitudinal and transverse waves in the elastic medium, λ and μ are the Lamé constants, ρ is the density of the elastic material, and $x = x_1/a$, $y = y_1/a$, and $t = c_2 t_1/a$ are dimensionless coordinates and time.

The fluid velocity potential $\varphi(x, y, t)$ satisfies the wave equation

$$\Delta\varphi - \beta_2^2 \frac{\partial^2 \varphi}{\partial t^2} = 0, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \varphi_1 = a c_2 \varphi, \qquad \beta_2 = \frac{c_2}{c_0}, \tag{3}$$

and the pressure p is defined by the Cauchy–Lagrange integral

$$p = -\rho_0 c_2^2 \frac{\partial \varphi}{\partial t} \tag{4}$$

(c_0 is the sound velocity in the fluid and ρ_0 is the fluid density).

Equations (1)–(3) are solved subject to the boundary conditions

$$\tau_{xy}(x, \pm h, t) = 0, \qquad -\infty < x < \infty; \tag{5}$$

$$\sigma_{yy}(x, h, t) = 0, \qquad -\infty < x < -1, \quad 1 < x < \infty; \tag{6}$$

$$\sigma_{yy}(x, -h, t) = -p(x, t), \qquad |x| < \infty; \tag{7}$$

$$v(x, h, t) = f(x, t), \qquad |x| < 1, \quad h = h_1/a; \tag{8}$$

$$\frac{\partial \varphi}{\partial y}(x, -h, t) = \frac{\partial v}{\partial t}(x, -h, t), \qquad |x| < \infty; \tag{9}$$

$$\varphi \to 0, \quad u \to 0, \quad v \to 0, \quad \tau_{xy} \to 0, \quad \sigma_{yy} \to 0 \quad \text{at} \quad x^2 + y^2 \to \infty \tag{10}$$

and initial conditions

$$u = v = \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \varphi = \frac{\partial \varphi}{\partial t} = 0 \quad \text{at} \quad t = 0.$$

In (5)–(10), τ_{xy} and σ_{yy} are the tangential and normal stresses in the elastic layer, respectively.

It is required to determine the distribution of normal contact stresses under the stamp $\sigma_{yy}(x, h, t) = -q(x, t)$, the stresses and displacements in the elastic layer, and the fluid velocity field.

Derivation of the Integral Equation. To solve Eqs. (1)–(3), we use the Laplace transform with respect to time

$$u^L(x, y, s) = \int_0^\infty u(x, y, t) \exp(-st)\, dt.$$

For the images, we obtain

$$\frac{\partial^2 u^L}{\partial x^2} + (1 - \beta_1^2) \frac{\partial^2 v^L}{\partial x\, \partial y} + \beta_1^2 \frac{\partial^2 u^L}{\partial y^2} = s^2 \beta_1^2 u^L; \tag{11}$$

$$\beta_1^2 \frac{\partial^2 v^L}{\partial x^2} + (1 - \beta_1^2) \frac{\partial^2 u^L}{\partial x\, \partial y} + \frac{\partial^2 v^L}{\partial y^2} = s^2 \beta_1^2 v^L; \tag{12}$$

$$\frac{\partial^2 \varphi^L}{\partial x^2} + \frac{\partial^2 \varphi^l}{\partial y^2} = s^2 \beta_2^2 \varphi^L.$$

For the stress images, we have

$$\sigma_{yy}^L = \lambda\Big(\frac{\partial u^L}{\partial x} + \frac{\partial v^L}{\partial y}\Big) + 2\mu \frac{\partial v^L}{\partial y}, \qquad \tau_{xy}^L = \mu\Big(\frac{\partial u^L}{\partial y} + \frac{\partial v^L}{\partial x}\Big).$$

For the images (11) and (12), the solution of the Lamé equations is sought in the form

$$\boldsymbol{u}^L = \nabla\Phi + \boldsymbol{u}_1.$$

These equations are valid if the new required functions satisfy the equations [6]

$$\operatorname{div} \boldsymbol{u}_1 = 0, \quad \Delta\Phi - \beta_1^2 s^2 \Phi = 0, \quad \Delta \boldsymbol{u}_1 - s^2 \boldsymbol{u}_1 = 0, \quad \boldsymbol{u}_1 = (u_1, v_1).$$

The solution of the image equations which is symmetric with respect to x is sought in the form

$$u_1 = \int_0^\infty \sin(\alpha x)\Big(A_1' \frac{\cosh(\gamma_2 y)}{\sinh(\gamma_2 h)} + C_1' \frac{\sinh(\gamma_2 y)}{\cosh(\gamma_2 h)}\Big)\, d\alpha,$$

$$v_1 = \int_0^\infty \cos(\alpha x)\Big(A_2' \frac{\sinh(\gamma_2 y)}{\sinh(\gamma_2 h)} + C_2' \frac{\cosh(\gamma_2 y)}{\cosh(\gamma_2 h)}\Big)\, d\alpha,$$

$$\Phi = \int_0^\infty \cos(\alpha x)\Big(A_3 \frac{\cosh(\gamma_1 y)}{\sinh(\gamma_1 h)} + C_3 \frac{\sinh(\gamma_1 y)}{\cosh(\gamma_1 h)}\Big)\, d\alpha, \quad \varphi^L = \int_0^\infty A(\alpha, s)\exp(\gamma_0 y)\cos(\alpha x)\, d\alpha,$$

where $\gamma_1^2 = \alpha^2 + \beta_1^2 s^2$, $\gamma_2^2 = \alpha^2 + s^2$, and $\gamma_0^2 = \alpha^2 + \beta_2^2 s^2$. Then, from the equation $\operatorname{div} \boldsymbol{u}_1 = 0$, we obtain

$$\alpha A_1' = \gamma_2 A_2', \qquad \alpha C_1' = -\gamma_2 C_2'.$$

Introducing the coefficients $\alpha A_2 = A_2'$ and $\alpha C_2 = C_2'$, from the boundary condition (5), we have

$$2\alpha\gamma_1 A_3 = -(\alpha^2 + \gamma_2^2) A_2, \quad 2\alpha\gamma_1 C_3 = -(\alpha^2 + \gamma_2^2) C_2, \quad A_1 = -\gamma_2 A_2, \quad C_1 = -\gamma_2 C_2.$$

For the transverse displacements at $y = \pm h$, we have

$$v^L(x, h, s) = -\frac{1}{2} s^2 \int_0^\infty \frac{\cos(\alpha x)}{\alpha} (A_2 + C_2)\, d\alpha,$$

$$v^L(x, -h, s) = \frac{1}{2} s^2 \int_0^\infty \frac{\cos(\alpha x)}{\alpha} (A_2 - C_2)\, d\alpha.$$

Similarly, for the normal stresses at $y = \pm h$, we obtain

$$\sigma_{yy}^L(x, h, s) = -\mu \int_0^\infty \cos(\alpha x)\Big(\frac{(\gamma_2^2 + \alpha^2)^2}{2\alpha\gamma_1} (A_2 \coth(\gamma_1 h) + C_2 \tanh(\gamma_1 h))$$
$$- 2\alpha\gamma_2 (A_2 \coth(\gamma_2 h) + C_2 \tanh(\gamma_2 h))\Big)\, d\alpha,$$

$$\sigma_{yy}^L(x, -h, s) = -\mu \int_0^\infty \cos(\alpha x)\Big(\frac{(\gamma_2^2 + \alpha^2)^2}{2\alpha\gamma_1} (A_2 \coth(\gamma_1 h) - C_2 \tanh(\gamma_1 h))$$
$$- 2\alpha\gamma_2 (A_2 \coth(\gamma_2 h) - C_2 \tanh(\gamma_2 h))\Big)\, d\alpha.$$

From boundary condition (9), we find that $A(\alpha, s) = s v^L / \gamma_0$. Then, in view of (4), $p^L(x, -h, s) = -\rho_0 c_2^2 s^2 v^L(x, -h, s)$.

Substituting the expressions obtained for σ_{yy}^L, p^L, and v^L into boundary condition (7) and introducing the notation

$$X_a = \frac{(\gamma_2^2 + \alpha^2)^2}{\gamma_1} \coth(\gamma_1 h) - 4\alpha^2 \gamma_2 \coth(\gamma_2 h), \qquad \rho_* = \frac{\rho_0}{\rho},$$

$$X_c = \frac{(\gamma_2^2 + \alpha^2)^2}{\gamma_1} \tanh(\gamma_1 h) - 4\alpha^2 \gamma_2 \tanh(\gamma_2 h), \qquad X_0 = \frac{\rho_* s^4}{\gamma_0},$$

we obtain the following equation for the coefficients A_2 and C_2:

$$(X_a + X_0) A_2 - (X_c + X_0) C_2 = 0 \tag{13}$$

(the quantity X_0 characterizes the effect of the fluid on the deformation of the elastic layer). For $y = h$, the boundary condition (6) for σ_{yy} is written as

$$\sigma_{yy}^L(x, h, s) = -q_*^L(x, s), \tag{14}$$

where $q_*^L(x, s) = q^L(x, s)$ at $|x| < 1$, $q_*^L(x, s) = 0$ at $|x| > 1$, and $q^L(x, s)$ is the L image of the required contact load under the stamp $q(x, t)$. Determining the Fourier transform of the function q_*^L

$$q_*^L(x, s) = \int_0^\infty Q^L(\alpha, s) \cos(\alpha x)\, d\alpha$$

and its inverse transform

$$Q^L(\alpha, s) = \frac{1}{\pi} \int_{-1}^{1} q^L(\xi, s) \cos(\alpha\xi)\, d\xi,$$

from the boundary condition (14), we obtain

$$X_a A_2 + X_c C_2 = 2Q^L \alpha. \tag{15}$$

Thus, for the coefficients A_2 and C_2, we have the system equations (13) and (15), from which we obtain

$$A_2 = 2\alpha Q^L(X_a + X_0)/D, \quad C_2 = 2\alpha Q^L(X_a + X_0)/D, \quad D = X_a(X_c + X_0) + X_c(X_a + X_0).$$

Substituting A_2 and C_2 into relation (14) and using $f^L(x, s)$ to denote the L-image of the function $f(x, t)$, we have

$$f^L(x, s) = -s^2 \int_0^\infty \cos(\alpha x) Q^L(\alpha, s) \frac{2X_0 + X_a + X_c}{D}\, d\alpha.$$

Using the expression for Q^L, we obtain the integral equation of the first kind for the function $q^L(\xi, s)$ ($|x| < 1$):

$$\int_{-1}^{1} q^L(\xi, s) k(x, \xi, s)\, d\xi = f^L(x, s)$$

with the kernel

$$k(x, \xi, s) = \frac{1}{\pi} s^2 \int_0^\infty \cos(\alpha\xi) \cos(\alpha x)\, \frac{2X_0 + X_a + X_c}{D}\, d\alpha.$$

For a flat stamp, $f^L(x, s) = f_0^L(s)$.

Some Properties of the Kernel $k(x, \xi, s)$. To study the properties of the kernel $k(x, \xi, s)$, we make the replacement $\alpha = sz$ in the improper integral and consider integrals of the form

$$J(s) = \int_0^\infty \cos(uz)\, K(s, z)\, dz, \quad u = s(x \pm \xi),$$

where $K(s, z) = (2X_0 + X_a + X_c)/D$. The function $K(s, z)$ contains the quantities

$$X_a = \frac{(2z^2 + 1)^2}{\gamma_1} \coth\,(sh\gamma_1) - 4z^2\gamma_2 \coth\,(sh\gamma_2),$$

$$X_c = \frac{(2z^2 + 1)^2}{\gamma_1} \tanh\,(sh\gamma_1) - 4z^2\gamma_2 \tanh\,(sh\gamma_2), \quad X_0 = \frac{\rho_*}{\gamma_0}$$

($\gamma_1^2 = z^2 + \beta_1^2$, $\gamma_2^2 = z^2 + 1$, and $\gamma_0^2 = z^2 + \beta_2^2$; for γ_0, γ_1, and γ_2, the former notation is kept). In view of $q^L(\xi, s) = q^L(-\xi, s)$, the kernel becomes

$$k(x, \xi, s) = -\frac{1}{\pi} \int_0^\infty \cos(sz)(x - \xi)\, K(s, z)\, dz,$$

where the Laplace transform parameter s enters only the arguments of the trigonometric and hyperbolic functions. To obtain an asymptotic solution for small values of s, in the expansions of $\coth x$ and $\tanh x$ we keep two and one term, respectively. Then, $X_a = Q(z)/(sh) + sh/3$ and $X_c = sh$ $(Q(z) = [1+4z^2(1-\beta_1^2)]/\gamma_1^2)$ and the function $K(s,z)$ can be written as

$$K(s,z) = a_0 + a_1 sh + a_2 s^2 h^2 + \dots \ ,$$

where $a_0 = 1/X_0$, $a_1 = 2/Q - 1/X_0^2$, and $a_2 = -a_1/X_0$. The integral $J(s)$ is written as the sum of the integrals $J_1(s)$ and $J_2(s)$:

$$J_1(s) = \int_0^1 \cos(uz)\, K(s,z)\, dz, \qquad J_2(s) = \int_1^\infty \cos(uz)\, K(s,z)\, dz.$$

To obtain an approximate expression for $J_1(s)$, we replace $\cos(uz)$ by the first two terms of its series expansion and $K(s,z)$ by its representation for small s. Then,

$$J_1(s) = q_0 + q_1 s + q_2 s^2,$$

where

$$q_0 = \int_0^1 a_0(z)\, dz, \quad q_1 = h \int_0^1 a_1(z)\, dz, \quad q_2 = -\frac{1}{2}(x-\xi)^2 \int_0^1 z^2 a_0(z)\, dz + h^2 \int_0^1 a_2(z)\, dz.$$

For large z, we assume that $\gamma_1 \approx z$ and $\gamma_2 \approx z$. Using the expansions

$$\coth\,(sh\gamma_1) \approx 1 + 2\sum_{m=1}^{\infty} \exp(-2mshz), \qquad \tanh\,(sh\gamma_1) \approx 1 + 2\sum_{m=1}^{\infty} (-1)^m \exp(-2mshz),$$

we obtain

$$X_a = X_{a\infty}[1 + 2\sum_{m=1}^{\infty} \exp(-2mshz)], \qquad X_c = X_{a\infty}[1 + 2\sum_{m=1}^{\infty} (-1)^m \exp(-2mshz)],$$

where

$$X_{a\infty} = (2z^2+1)^2/\sqrt{z^2+\beta_1^2} - 4z^2\sqrt{z^2+1}.$$

In this case ($|z| \gg 1$), the following representation holds:

$$K(s,z) = b_0 + b_1 \exp(-2shz) + b_2 \exp(-4shz) + \dots \ ,$$

where $b_0 = 1/X_{a\infty}$, $b_1 = 0$, and $b_2 = 2(1 - X_0/X_{a\infty})/(X_0 + X_{a\infty})$.
For large z,

$$X_{a\infty} = 2(1-\beta_1^2)z + \frac{h_a}{z}, \quad h_a = \frac{1}{2}(3 + 3\beta_1^4 - 4\beta_1^2), \quad X_0 = \rho_*\Big(\frac{1}{z} - \frac{\beta_2^2}{2z^3}\Big);$$

therefore,

$$b_0 = \frac{d_1}{z} - \frac{h_a d_1^2}{z^3}, \quad d_1 = \frac{1}{2}\,\frac{1}{1-\beta_1^2}, \quad b_2 = \frac{2d_1}{z} - \frac{2d_1^2(h_a + 2\rho_*)}{z^3}.$$

Introducing the variable $u = s|x-\xi|$ and substituting $K(s,z)$ into $J_2(s)$ for large z, we obtain

$$J_2(s) = \int_1^\infty \cos(uz)\Big(\frac{d_1}{z} - \frac{h_a d_1^2}{z^3} + \dots\Big)\, dz + 2\int_1^\infty \cos(uz)\Big(\frac{d_1}{z} - \frac{d_1^2(h_a + 2\rho_*)}{z^3}\Big) \exp(-4hsz)\, dz.$$

Introducing the notation

$$I_1 = \int_1^\infty \frac{\cos(uz)}{z}\, dz = -\mathrm{Ci}(u), \qquad I_2 = \int_1^\infty \frac{\cos(uz)}{z^3}\, dz,$$

$$I_3 = \int_1^\infty \frac{\cos(uz)}{z} \exp(-4hsz)\, dz, \qquad I_4 = \int_1^\infty \frac{\cos(uz)}{z^3} \exp(-4hsz)\, dz$$

[Ci(u) is the integral cosine]. Then, for small s, the asymptotic form of the integrals I_k ($k = 1, \ldots, 4$) is written as

$$I_1 = -C - \ln u + u^2/4, \quad I_2 = (1 + u^2 \ln u + u^2(C - 3/2))/2,$$

$$I_3 = -C - \ln(s\sqrt{16h^2 + (x-\xi)^2}\,) + 4sh + s^2\alpha_1/4, \quad I_4 = 1/2 - 4sh + \alpha_1 s^2 \ln s + \alpha_2 s^2,$$

where C is the Euler constant,

$$\alpha_1 = \frac{1}{2}\,[(x-\xi)^2 - 16h^2], \quad \alpha_2 = \alpha_1\Big(C - \frac{3}{2} + \frac{1}{2}\ln[16h^2 + (x-\xi)^2]\Big) + 4h(x-\xi)\arctan\frac{x-\xi}{4h}.$$

Using the obtained asymptotic expressions of the integrals for small values of s, we write the integral equation for a flat stamp in the form

$$\frac{1}{\pi}\int_{-1}^{1} q^L(\xi, s)\ln(s|x-\xi|)\, d\xi = g^L(x, s), \tag{16}$$

where

$$g^L(x, s) = -\frac{1}{\pi}\int_{-1}^{1} q^L(\xi, s)\ln[16h^2 + (x-\xi)^2]\, d\xi + f^L(s) + \int_{-1}^{1} q^L(\xi, s)F(x, \xi, s)\, d\xi,$$

$$F = a_{0*} + a_{1*}s + a_{2*}s^2\ln s + a_{3*}s^2, \tag{17}$$

$$\pi d_1 a_{0*} = q_0 - 3Cd_1 - d_1^2(3h_a/2 + 2\rho_*), \qquad \pi d_1 a_{1*} = q_1 + 8d_1 h + 8hd_1^2(h_a + 2\rho_*),$$

$$\pi d_1 a_{2*} = -h_a d_1^2(x-\xi)^2/2 - 2d_1\alpha_1(h_a + 2\rho_*),$$

$$\pi d_1 a_{3*} = q_2 + d_1(x-\xi)^2/4 - (1/2)h_a d_1^2(x-\xi)^2\ln|x-\xi| + d_1\alpha_1 - 2d_1^2\alpha_2(h_a + 2\rho_*).$$

Equation (16) for $q^L(\xi, s)$ is an equation of the first kind with a logarithmic kernel [7]. The second term containing logarithm takes into account the presence of the boundary of the layer located at a distance $2h$ from the stamp, and it is similar to the term calculated, for example, of the flow over a profile at a distance $2h$ from a rigid plate using the reflection method [8].

Differentiation of Eq. (16) with respect to x yields the singular integral equation

$$\frac{1}{\pi}\int_{-1}^{1} q^L(\xi, s)\,\frac{d\xi}{x-\xi} = -\frac{2}{\pi}\int_{-1}^{1} q^L(\xi, s)\,\frac{(x-\xi)\, d\xi}{16h^2 + (x-\xi)^2} + \int_{-1}^{1} q^L(\xi, s)\,\frac{\partial F}{\partial x}(x, \xi, s)\, d\xi.$$

Assuming temporarily that the right side of $\partial g^L/\partial x$ is known, we write the solution of this equation in the form

$$q^L(x, s) = \frac{P^L(s)}{\pi\sqrt{1-x^2}} + \frac{1}{\pi\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-\eta^2}}{\eta - x}\,\frac{\partial g^L}{\partial \eta}(\eta, s)\, d\eta, \tag{18}$$

where

$$P^L(s) = \int_{-1}^{1} q^L(\xi, s)\, d\xi, \qquad \frac{\partial F}{\partial \eta} = \frac{\partial a_{2*}}{\partial \eta}\, s^2\ln s + \frac{\partial a_{3*}}{\partial \eta}\, s^2,$$

$$\pi\,\frac{\partial a_{2*}}{\partial \eta} = -d_1(\eta - \xi)(3h_a + 2\rho_*),$$

$$\pi d_1 \frac{\partial a_{3*}}{\partial \eta} = -(\eta - \xi) \int_0^1 z^2 a_0(z)\, dz + \frac{3}{2} d_1(\eta - \xi) - h_a d_1^2(\eta - \xi) \ln|\eta - \xi|$$

$$- \frac{1}{2} h_a d_1^2(\eta - \xi) - 2d_1^2(h_a + 2\rho_*) \frac{\partial \alpha_2}{\partial \eta},$$

$$\frac{\partial \alpha_2}{\partial \eta} = \Big(C - \frac{3}{2} + \frac{1}{2} \ln\big[16h^2 + (\eta - \xi)^2\big]\Big)(\eta - \xi) + \frac{(\eta - \xi)^3}{16h^2 + (\eta - \xi)^2} + 4h \arctan \frac{\eta - \xi}{4h}.$$

The required function is represented as the series

$$q^L(x, s) = P^L(s)(Q_0 + Q_1 s^2 \ln s + Q_2 s^2 + \dots). \tag{19}$$

Substituting (19) into Eq. (18) and collecting terms of the same powers of s, we obtain the following integral equations for Q_0, Q_1, and Q_2:

$$Q_0(x) = \frac{1}{\pi\sqrt{1 - x^2}} \Big(1 - \frac{2}{\pi} \int_{-1}^{1} \frac{\sqrt{1 - \eta^2}}{\eta - x}\, d\eta \int_{-1}^{1} Q_0(\xi) \frac{\eta - \xi}{16h^2 + (\eta - \xi)^2}\, d\xi\Big); \tag{20}$$

$$Q_1(x) = -\frac{2}{\pi^2\sqrt{1 - x^2}} \int_{-1}^{1} \frac{\sqrt{1 - \eta^2}}{\eta - x}\, d\eta \int_{-1}^{1} Q_1(\xi) \frac{\eta - \xi}{16h^2 + (\eta - \xi)^2}\, d\xi$$

$$+ \frac{1}{\pi\sqrt{1 - x^2}} \int_{-1}^{1} \frac{\sqrt{1 - \eta^2}}{\eta - x}\, d\eta \int_{-1}^{1} Q_0(\xi) \frac{\partial a_{2*}}{\partial \eta}\, d\xi; \tag{21}$$

$$Q_2(x) = -\frac{2}{\pi^2\sqrt{1 - x^2}} \int_{-1}^{1} \frac{\sqrt{1 - \eta^2}}{\eta - x}\, d\eta \int_{-1}^{1} Q_2(\xi) \frac{\eta - \xi}{16h^2 + (\eta - \xi)^2}\, d\xi$$

$$+ \frac{1}{\pi\sqrt{1 - x^2}} \int_{-1}^{1} \frac{\sqrt{1 - \eta^2}}{\eta - x}\, d\eta \int_{-1}^{1} Q_0(\xi) \frac{\partial a_{3*}}{\partial \eta}\, d\xi. \tag{22}$$

To find the functions $Q_0(x)$, $Q_1(x)$, and $Q_2(x)$, we expand the integrand functions on the right side of Eqs. (20)–(22) in series in a certain parameter related to the layer thickness h. This parameter is taken to be $\tau = \sqrt{1 + h^2/4} - h/2$; moreover, it is assumed that $\tau < 1$ for any h, in particular, $\tau = 0$ for $h = \infty$ and $\tau = 1$ for $h = 0$. The following expansion holds:

$$\frac{1}{(x - \xi)^2 + 16h^2} = \frac{\tau^2}{16} \Big(1 - \tau^2(\tau^2 - 2) - \frac{\tau^2(x - \xi)^2}{16} + \dots\Big). \tag{23}$$

The required functions can be represented as series in τ:

$$Q_n = Q_{n0} + \tau^2 Q_{n1} + \tau^4 Q_{n2} + \dots \qquad (n = 0, 1, 2).$$

Substituting these series into Eqs. (20)–(22) and equating terms of the same powers of τ, we obtain

$$Q_{n0}(x) = \frac{1}{\pi\sqrt{1 - x^2}}, \quad Q_{01}(x) = -\frac{1}{8\pi^2\sqrt{1 - x^2}} \int_{-1}^{1} \frac{\sqrt{1 - \eta^2}}{\eta - x}\, d\eta \int_{-1}^{1} Q_{00}(\xi)(\eta - \xi)\, d\xi,$$

$$Q_{02}(x) = -\frac{1}{8\pi^2\sqrt{1 - x^2}} \int_{-1}^{1} \frac{\sqrt{1 - \eta^2}}{\eta - x}\, d\eta \int_{-1}^{1} (\eta - \xi)\Big[Q_{00}(\xi)\Big(2 - \frac{(\eta - \xi)^2}{16}\Big) + Q_{01}(\xi)\Big]\, d\xi.$$

Calculations of the integrals on the right sides of the expressions for $Q_{01}(x)$ and $Q_{02}(x)$ yields

$$Q_{01}(x) = -\frac{x^2}{4\pi\sqrt{1-x^2}}, \quad Q_{02}(x) = -\frac{1}{128\pi\sqrt{1-x^2}}\,(x^4 + 56.5x^2 - 0.125).$$

A similar method was used in [8, 9].

Passing to Laplace originals in (19) for the case of a flat stamp considered, we obtain the following asymptotic expression for the contact load under the stamp for large times:

$$q(x,t) = Q_0(x)P(t) + \frac{d^2P(t)}{dt^2}\,[Q_2(x) - CQ_1(x)] - Q_1(x)\,\frac{d}{dt}\int\limits_0^t \frac{d^2P}{d\tau^2}\ln(t-\tau)\,d\tau.$$

The Laplace original of the expression $s^2P^L(s)\ln s$ was obtained using the convolution theorem and the relation $s^{-1}\ln s \Rightarrow -\ln t - C$ [10].

To find $P^L(s)$, we multiply Eq. (16) by $1/\sqrt{1-x^2}$ and integrate the result over x from -1 to 1:

$$\frac{1}{\pi}\int\limits_{-1}^{1} q^L(\xi,s)d\xi \int\limits_{-1}^{1} \frac{\ln s|x-\xi|}{\sqrt{1-x^2}}\,dx - \int\limits_{-1}^{1} q^L(\xi,s)d\xi \int\limits_{-1}^{1} \frac{F(x,\xi,s)}{\sqrt{1-x^2}}\,dx$$

$$+\frac{2}{\pi}\int\limits_{-1}^{1} q^L(\xi,s)\,d\xi \int\limits_{-1}^{1} \frac{\ln s\sqrt{16h^2+(x-\xi)^2}}{\sqrt{1-x^2}}\,dx = \pi f^L(s). \quad (24)$$

On the left side of expression (24), the first integral over x has the form [11]

$$\int\limits_{-1}^{1} \frac{\ln s|x-\xi|}{\sqrt{1-x^2}}\,dx = \pi\ln\frac{s}{2}.$$

The second integral is written as

$$\int\limits_{-1}^{1} \frac{\ln s\sqrt{16h^2+(x-\xi)^2}}{\sqrt{1-x^2}}\,dx = \pi\ln s + I_2(\xi),$$

where

$$I_2(\xi) = \int\limits_{-1}^{1} \frac{\ln\sqrt{16h^2+(x-\xi)^2}}{\sqrt{1-x^2}}\,dx.$$

Differentiating $I_2(\xi)$ over ξ and using expansion (23) in the parameter τ, we obtain

$$\frac{dI_2}{d\xi} = \frac{\tau^2}{16}\,\pi\xi - \frac{\tau^4\pi}{256}\,\xi(\xi^2 - 30.5).$$

Integration of this expression with respect to ξ yields

$$I_2(\xi) = \frac{\tau^2}{32}\,\pi\xi^2 + \frac{\tau^4\pi}{1024}\,\xi^2(61-\xi^2) + C_*.$$

The constant C_* is determined as the value of the integral I_2 for $\xi = 0$ [11]:

$$C_* = \pi\ln(4h) + \pi\ln 0.5\Big[1 + \sqrt{1 + 1/(16h^2)}\Big].$$

Substitution of the values of the integrals and the function $F(x,\xi,s)$ defined by formula (17) into Eqs. (24) yields

$$P^L(s) = \pi f^L(s)/w(s), \quad (25)$$

where

$$w(s) = d_1\ln s + d_0 + d_2 s + d_3 s^2\ln s + d_4 s^2 + \dots\ ,$$

$$d_0 = -(\ln 2 + \pi a_{0*})J_0 + \frac{2}{\pi}\int_{-1}^{1} Q_0(\xi) I_2(\xi)\, d\xi, \quad d_1 = 3J_0, \quad d_2 = -\pi a_{1*} J_0,$$

$$d_3 = -(\ln 2 + \pi a_{0*})J_1 + 3J_2 + \frac{2}{\pi}\int_{-1}^{1} Q_1(\xi) I_2(\xi)\, d\xi - J_0 \int_{-1}^{1} \frac{a_{2*}(x)}{\sqrt{1-x^2}}\, dx,$$

$$d_4 = -(\ln 2 + \pi a_{0*})J_2 + \frac{2}{\pi}\int_{-1}^{1} Q_2(\xi) I_2(\xi)\, d\xi - J_0 \int_{-1}^{1} \frac{a_{3*}(x)}{\sqrt{1-x^2}}\, dx,$$

$$J_k = \int_{-1}^{1} Q_k(\xi)\, d\xi, \quad k = 0, 1, 2.$$

To obtain asymptotic formulas for $P^L(s)$ for small s, we write $w^{-1}(s)$ in (25) in the form

$$w^{-1}(s) = \frac{1}{d_1 \ln s}\Big(1 + \frac{d_0}{d_1 \ln s} + \frac{d_2 s}{d_1 \ln s} + \ldots\Big)^{-1},$$

and expand the expression in parentheses in a series taking into account that $s < 1$. Then,

$$P^L(s) = \frac{\pi}{d_1} f^L(s) g_1^L(s) - \frac{\pi d_0}{d_1^2} f^L(s) g_2^L(s) + \ldots\ ,$$

where $g_1^L(s) = 1/\ln s$ and $g_2^L = 1/\ln^2 s$. Using the convolution theorem, we pass to the Laplace originals

$$P(t) = \frac{\pi}{d_1}\int_0^t f(t-\tau) g_1(\tau)\, d\tau - \frac{\pi d_0}{d_1^2}\int_0^t f(t-\tau) g_2(\tau)\, d\tau + \ldots\ , \qquad (26)$$

where, according to [10],

$$g_1(\tau) = \int_0^\infty \frac{\tau^{z-1}}{\Gamma(z)}\, dz, \qquad g_2(\tau) = \int_0^\infty \frac{z\tau^{z-1}}{\Gamma(z)}\, dz,$$

[$\Gamma(z)$ is a gamma function].

As $t \to \infty$, the integrals over τ become improper. To estimate the convergence of these integrals, we study the behavior of the functions $g_1(\tau)$ and $g_2(\tau)$ for large values of the argument. Applying the saddle-point method [12] to estimate the integrals $g_1(\tau)$ and $g_2(\tau)$ as $\tau \to \infty$, we write them as

$$g_1(\tau) = \int_0^\infty \exp\left(h_1(z,\tau)\right) dz, \qquad g_2(\tau) = \int_0^\infty \exp\left(h_2(z,\tau)\right) dz,$$

where $h_1(z,\tau) = (z-1)\ln\tau - \ln\Gamma(z)$ and $h_2(z,\tau) = (z-1)\ln\tau + \ln z - \ln\Gamma(z)$. For the saddle point, we have $h_i'(z) = 0$. Since, for large z ($z \sim t$, $\tau \gg 1$) $\ln\Gamma(z) \sim z\ln z - z + \ldots$, we find the saddle point $z = \tau$ at which $h''(\tau,\tau) = -1/\tau$. The contribution to the integral $g_1(\tau)$ is given by

$$V_{1c}(\tau) = \sqrt{2\pi\tau}\,\tau^{\tau-1}\Gamma^{-1}(\tau) = e^\tau,$$

because $\Gamma(\tau) = \sqrt{2\pi}\exp(-\tau)\tau^{\tau-1/2}$ as $\tau \to \infty$. For the integral $g_2(\tau)$, the saddle point $z = \tau$, and, for $\tau \to \infty$, its contribution to the integral is $V_{2c}(\tau) = \tau e^\tau$.

Thus, for the convergence of the integrals in the representation (26) for $P(t)$, it is necessary that the given stamp displacement function $f(x,t)$ increase exponentially as $t \to \infty$. In this case, the total load on the stamp also increases exponentially with time, because the expression for $P(t)$ contains the integrand function $f(t-\tau)$.

In [13], a similar distribution was obtained by asymptotic analysis for the problem of pure shear of an elastic layer with a fixed foundation by a stamp acted upon by a shear force for a large time. In addition, in [13], it is noted that if the displacement vector is sought in the form $\boldsymbol{u}(x,y,t) = \boldsymbol{u}_0(x,y)\exp(\nu t)$, asymptotic analysis makes it possible to construct an asymptotic solution of the problem for large times without applying the Laplace transform.

The function $f(t)$ obtained from the condition of convergence of the integrals in (26), is a particular case of the law of motion of the stamp. Generally, these integrals can be divergent. This is due to the fact that in a number of two-dimensional contact problems, the passage to the limit in the parameter cannot be performed. In particular, it has been shown [14] that a number of two-dimensional mixed problems for a layer of thickness λ are reduced to integral equations of the first kind, which, for large λ, can be written as

$$\int_{-1}^{1} q(\xi)(-\ln|\xi - x| + d)\,d\xi = f(x),$$

where $d = \ln(4\lambda/\pi)$. For large λ, the kernel of this equation tends to infinity. In [14], it is noted that, because of the presence of the constant d in the kernel, the limiting case $\lambda = \infty$ cannot be considered. In [14], this is regarded as a consequence of the replacement of three-dimensional problems by two-dimensional ones. The kernel of the integral equation (16) also contains a logarithmic constant, which tends to infinity as $s \to 0$; therefore, the solution of this equation tends to infinity as $\ln s$, i.e., as $s \to 0$, the passage to the limit is absent from the solution of the steady-state problem. In [3], it is also noted that, for $t \to \infty$, this passage to the limit cannot be performed.

In two-dimensional problems of elasticity and hydrodynamics, logarithmic terms appear in the case where the examined region of a continuous medium contains an infinitely distant point. This, however, does not prevent the use of the obtained solutions in the part of the region, in which the corresponding quantities are small (for example, displacements in elastic theory or the potential in hydrodynamics). The solution obtained can be used for finite times of interaction, similarly to asymptotic solutions in two-dimensional contact problems for layers of finite values of the parameter λ.

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